Denotational Semantics

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Recap: Denotational semantics

- a mapping of PCF types τ to domains $[\![\tau]\!]$;
- a mapping of PCF contexts Γ to domains $[\![\Gamma]\!]$;
- a mapping of closed, well-typed PCF terms $\cdot \vdash t : \tau$ to elements $[\![t]\!] \in [\![\tau]\!]$;
- a mapping of arbitrary terms $\Gamma \vdash t : \tau$ to continuous functions $[\![\Gamma]\!] \to [\![\tau]\!]$

Compositionality: $[\![t]\!] = [\![t']\!] \Rightarrow [\![\mathcal{C}[t]]\!] = [\![\mathcal{C}[t']]\!]$.

Soundness: for any type τ, $t \Downarrow_\tau v \Rightarrow [\![t]\!] = [\![v]\!]$.

Adequacy: for $\gamma = \texttt{bool}$ or $\texttt{nat}$, if $t \in \text{PCF}_\gamma$ and $[\![t]\!] = [\![v]\!]$ then $t \Downarrow_\gamma v$.

Recap: Types and contexts

$$[\![\texttt{nat}]\!] \stackrel{\text{def}}{=} \mathbb{N}_\bot \qquad \text{(flat domain)}$$

$$[\![\texttt{bool}]\!] \stackrel{\text{def}}{=} \mathbb{B}_\bot \qquad \text{(flat domain)}$$

$$[\![\tau \to \tau']\!] \stackrel{\text{def}}{=} [\![\tau]\!] \to [\![\tau']\!] \qquad \text{(function domain)}$$

$$[\![\cdot]\!] = \mathbb{1} \qquad \text{(one element set)}$$

$$[\![\Gamma, x{:}\,\tau]\!] = [\![\Gamma]\!] \times [\![\tau]\!] \qquad \text{(product domain)}$$

Recap: Terms

$$
\begin{aligned}
[\![\texttt{0}]\!] &= \lambda\, \rho \in [\![\Gamma]\!].\ 0 \\
[\![\texttt{true}]\!] &= \lambda\, \rho \in [\![\Gamma]\!].\ \text{true} \\
[\![\texttt{false}]\!] &= \lambda\, \rho \in [\![\Gamma]\!].\ \text{false} \\
[\![\texttt{succ}\,(t)]\!] &= \text{succ}_\perp \circ [\![t]\!] \\
[\![\texttt{pred}\,(t)]\!] &= \text{pred}_\perp \circ [\![t]\!] \\
[\![\texttt{zero?}\,(t)]\!] &= \text{zero?}_\perp \circ [\![t]\!]
\end{aligned}
$$

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\begin{aligned}
[\![\texttt{if}\ b\ \texttt{then}\ t\ \texttt{else}\ t']\!] &= \text{if} \circ \langle [\![b]\!], \langle [\![t]\!], [\![t']\!] \rangle \rangle \\
[\![x]\!] &= \pi_x \\
[\![t_1\ t_2]\!] &= \text{eval} \circ \langle [\![t_1]\!], [\![t_2]\!] \rangle \\
[\![\texttt{fun}\ x{:}\tau.\ t]\!] &= \text{cur}([\![t]\!]) \\
[\![\texttt{fix}\ f]\!] &= \text{fix} \circ [\![f]\!]
\end{aligned}
$$

Recap: Evaluation contexts and compositionality

We define the denotation for an evaluation contexts $\Gamma \vdash_{\Delta,\sigma} \mathcal{C} : \tau$ inductively to be a continuous function

$$[\![\mathcal{C}]\!] : ([\![\Delta]\!] \to [\![\sigma]\!]) \to [\![\Gamma]\!] \to [\![\tau]\!]$$

We show by induction on the typing derivation of $\mathcal{C}$ that:

$$[\![\mathcal{C}[t]]\!] = [\![\mathcal{C}]\!]([\![t]\!])$$

This gives us compositionality for free:

$$[\![t]\!] = [\![t']\!] \Rightarrow [\![\mathcal{C}[t]]\!] = [\![\mathcal{C}[t']]\!]$$

Substitution property of the semantic function

Assume

$$\Gamma \vdash u : \sigma$$
$$\Gamma, x{:}\sigma \vdash t : \tau$$

Then for all $\rho \in [\![\Gamma]\!]$

$$[\![t[u/x]]\!](\rho) = [\![t]\!](\rho[x \mapsto [\![u]\!](\rho)]).$$

In particular when $\Gamma = \cdot$, $[\![t]\!] : [\![\sigma]\!] \to [\![\tau]\!]$ and

$$[\![t[u/x]]\!] = [\![t]\!]([\![u]\!])$$

Soundness

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Theorem

For all PCF types τ and all closed terms $t, v \in \mathrm{PCF}_\tau$ with v a value, if $t \Downarrow_\tau v$, then

$$[\![t]\!] = [\![v]\!] \in [\![\tau]\!]$$

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We prove by rule induction on $t \Downarrow_\tau v$ that

$$\Phi(t, \tau, v) \overset{\text{def}}{\Leftrightarrow} [\![t]\!] = [\![v]\!] \in [\![\tau]\!]$$

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Corollary

If $\cdot \vdash t : \mathtt{nat}$ and $[\![t]\!] = \bot$ then $t \Uparrow_{\mathtt{nat}}$. Similarly for $\mathtt{bool}$, but not for $\tau \to \sigma$!

PCF EVALUATION

$$\textsc{Val}\ \frac{\vdash v : \tau}{v \Downarrow_\tau v}$$

PCF evaluation

$$\textsc{Val}\ \frac{\vdash v : \tau}{v \Downarrow_\tau v} \qquad \textsc{Succ}\ \frac{t \Downarrow_{\mathsf{nat}} v}{\mathtt{succ}\,(t) \Downarrow_{\mathsf{nat}} \mathtt{succ}\,(v)} \qquad \textsc{Pred}\ \frac{t \Downarrow_{\mathsf{nat}} \mathtt{succ}\,(v)}{\mathtt{pred}\,(t) \Downarrow_{\mathsf{nat}} v}$$

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$$\textsc{ZeroZ}\ \frac{t \Downarrow_{\mathtt{nat}} \mathtt{0}}{\mathtt{zero?}\,(t) \Downarrow_{\mathtt{bool}} \mathtt{true}} \qquad \textsc{ZeroS}\ \frac{t \Downarrow_{\mathtt{nat}} \mathtt{succ}\,(v)}{\mathtt{zero?}\,(t) \Downarrow_{\mathtt{bool}} \mathtt{false}}$$

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$$\textsc{IfT}\ \frac{b \Downarrow_{\mathtt{bool}} \mathtt{true} \qquad t_1 \Downarrow_\tau v}{\mathtt{if}\ b\ \mathtt{then}\ t_1\ \mathtt{else}\ t_2 \Downarrow_\tau v} \qquad \textsc{IfF}\ \frac{b \Downarrow_{\mathtt{bool}} \mathtt{false} \qquad t_2 \Downarrow_\tau v}{\mathtt{if}\ b\ \mathtt{then}\ t_1\ \mathtt{else}\ t_2 \Downarrow_\tau v}$$

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$$\textsc{Fun}\ \frac{t \Downarrow_{\sigma\to\tau} \mathtt{fun}\,x{:}\sigma.\,t' \qquad t'[u/x] \Downarrow_\tau v}{t\,u \Downarrow_\tau v} \qquad \textsc{Fix}\ \frac{t\,(\mathtt{fix}(t)) \Downarrow_\tau v}{\mathtt{fix}(t) \Downarrow_\tau v}$$

Adequacy

Soundness and adequacy

Proposition (Soundness)

For all PCF types τ and all closed $t, v \in \mathrm{PCF}_\tau$ with v a value, if $t \Downarrow_\tau v$ is derivable, then

$$[\![t]\!] = [\![v]\!] \in [\![\tau]\!]$$

Proposition (Adequacy)

For any closed PCF term t and value v of ground type $\gamma \in \{\mathtt{nat}, \mathtt{bool}\}$

$$[\![t]\!] = [\![v]\!] \Rightarrow t \Downarrow_\gamma v$$

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$$[\![\mathtt{fun}\, x{:}\tau.\, (\mathtt{fun}\, y{:}\tau.\, y)\, x]\!] = [\![\mathtt{fun}\, x{:}\tau.\, x]\!] : [\![\tau]\!] \to [\![\tau]\!]$$
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A more serious example for $\Gamma = (f : \texttt{nat} \to \texttt{nat})$

$$[\![\texttt{fun}\, x{:}\texttt{nat}.\, (\texttt{if zero?}\, (f\ x)\ \texttt{then true else true})]\!]$$
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$$\stackrel{?}{=} [\![\mathtt{fun}\, x{:}\mathtt{nat}.\, \mathtt{true}]\!]$$

This denotational equality holds exactly when f is a total function. But there is no hope that we can decide what the first expression should evaluate to: this would mean solving the halting problem for f!

Adequacy

Formal approximation

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The same proof shows adequacy for `bool`.

Formal approximation at base types

We define the formal approximation relation recursively on the type τ:

$$\lhd_\tau \subseteq [\![\tau]\!] \times \mathrm{PCF}_\tau$$

On base types, we let:

$$\begin{aligned} d \lhd_{\mathtt{nat}} t &\overset{\text{def}}{\Leftrightarrow} (d \in \mathbb{N} \Rightarrow t \Downarrow_{\mathtt{nat}} \underline{d}) \\ d \lhd_{\mathtt{bool}} t &\overset{\text{def}}{\Leftrightarrow} (d = \mathrm{true} \Rightarrow t \Downarrow_{\mathtt{bool}} \mathtt{true}) \\ &\quad \wedge (d = \mathrm{false} \Rightarrow t \Downarrow_{\mathtt{bool}} \mathtt{false}) \end{aligned}$$

- Exactly what we asked for in the previous slide!
- Note though that $\bot \lhd_{\mathtt{nat}} t$ for all $t \in \mathrm{PCF}_{\mathtt{nat}}$.

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By definition! We let:

$$d \lhd_{\tau \to \tau'} t \overset{\text{def}}{\Leftrightarrow} \forall e \in [\![\tau]\!].\ \forall u \in \textsc{Pcf}_\tau\ .\ (e \lhd_\tau u \Rightarrow d(e) \lhd_{\tau'} t\ u)$$

Formal approximation for open terms

$$\textsc{Abs}\ \frac{\Gamma, x{:}\tau \vdash t : \tau'}{\Gamma \vdash \mathsf{fun}\, x{:}\tau.\, t : \tau \to \tau'}$$

To prove the fundamental property, we also need to talk about open terms.

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$$[\![t]\!]([\![u]\!]) = [\![(t[u/x])]\!] \qquad \text{Semantic application} \approx \text{syntactic substitution}$$

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Parallel substitution: $\cdot \vdash \sigma : \Gamma$ assigns to each $x \in \mathrm{dom}(\Gamma)$ a term $\sigma(x) \in \mathrm{PCF}_{\Gamma(x)}$
We define also for $\rho \in [\![\Gamma]\!]$:

$$\rho \lhd_\Gamma \sigma \overset{\text{def}}{\Leftrightarrow} \forall x \in \mathrm{dom}(\Gamma), \rho(x) \lhd_{\Gamma(x)} \sigma(x)$$

For any

- context Γ and type τ
- term t such that $\Gamma \vdash t : \tau$
- environment $\rho \in [\![\Gamma]\!]$
- substitution $\cdot \vdash \sigma : \Gamma$

we have that

$$\rho \lhd_\Gamma \sigma \qquad \Rightarrow \qquad [\![t]\!](\rho) \lhd_\tau t[\sigma].$$

Corollary

For every term $t \in \mathrm{PCF}_\tau$, we have $[\![t]\!] \lhd_\tau t$.